

Axisymmetric Dynamo Solutions

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It is shown that a small anisotropy of the magnetic diffusivity tensor admits stationary axisymmetric solutions of the kinetic dynamo problem. For the case of a large aspect ratio the solutions can be expressed explicitly in terms of elementary functions.

Introduction

In the incompressible case the stationary kinematic dynamo problem is described by the following equations:

equation of continuity for the velocity \mathbf{v} :

$$\operatorname{div} \mathbf{v} = 0; \quad (1)$$

Maxwell's equation for the magnetic field \mathbf{B} :

$$\operatorname{div} \mathbf{B} = 0; \quad (2)$$

induction equation in a frame moving with non-relativistic velocity \mathbf{v} :

$$\eta \operatorname{curl} \mathbf{B} + \nabla U - \mathbf{v} \times \mathbf{B} = 0. \quad (3)$$

In addition to the fields \mathbf{v} , \mathbf{B} , the electric potential U must be single-valued in space in order that the configuration be self-sustained.

Equations (1)–(3) are valid in the conducting fluid. Outside the conductor there is a vacuum governed by the equations

$$\operatorname{curl} \mathbf{B} = \operatorname{div} \mathbf{B} = \Delta U = 0. \quad (4)$$

If the vacuum and fluid are separated by a discontinuity surface S , then the fields have to satisfy

$$\mathbf{B}, U \text{ continuous across } S \quad (5)$$

while the magnetic and electric fields should tend to zero for large distances from the conductor:

$$\mathbf{B}, \nabla U \rightarrow 0, \quad \mathbf{r} \rightarrow \infty. \quad (6)$$

If the magnetic diffusivity η is a scalar, then Cowling's theorem [1] states that (1)–(6) do not admit axisymmetric poloidal solutions. This result was generalized

to the poloidal and toroidal axisymmetric case in [2]. Further generalization was achieved in [3], where it was shown for a time-dependent solenoidal flow that all axisymmetric magnetic fields decay in time. The first non-existence proof for a non-solenoidal flow was given in [4] for the stationary case. The magnetic field of the earth and of Saturn are very close to axisymmetry, however. Thus it has been speculated in [5] that compressibility and time dependence together could allow axisymmetric fields to grow. In [6]–[11] it was shown that this is impossible.

In the following axisymmetric solutions of (1)–(6) are considered for the case where the diffusivity η is not a scalar but an axisymmetric tensor.

Axisymmetric Solutions

Let us introduce cylindrical coordinates ϱ, θ, ζ . The general solenoidal axisymmetric field has the form

$$\mathbf{B} = \nabla \theta \times \nabla F + f \nabla \theta = \frac{1}{\varrho} \begin{pmatrix} F_\zeta \\ f \\ -F_\varrho \end{pmatrix}, \quad (7)$$

where $F, f(\varrho, \zeta)$ are arbitrary axisymmetric scalars and the subscripts denote partial derivatives. Let the respective representation for the velocity field be

$$\mathbf{v} = \nabla \theta \times \nabla G + g \nabla \theta = \frac{1}{\varrho} \begin{pmatrix} G_\zeta \\ g \\ -G_\varrho \end{pmatrix}, \quad G, g(\varrho, \zeta). \quad (8)$$

This yields

$$\mathbf{v} \times \mathbf{B} = \frac{1}{\varrho^2} \begin{pmatrix} G_\varrho f - F_\varrho g \\ G_\zeta F_\varrho - G_\varrho F_\zeta \\ G_\zeta f - F_\zeta g \end{pmatrix}, \quad (9)$$

and the current density is proportional to

$$\operatorname{curl} \mathbf{B} = \nabla f \times \nabla \theta + (\Delta_* F) \nabla \theta = \frac{1}{\varrho} \begin{pmatrix} -f_\zeta \\ \Delta_* F \\ f_\varrho \end{pmatrix}, \quad (10)$$

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where

$$\begin{aligned}\Delta_* &= \frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + \frac{\partial^2}{\partial \zeta^2} \\ &= \varrho^2 \nabla \cdot \frac{\nabla}{\varrho^2} = \nabla \cdot \left(\nabla - 2 \frac{\nabla \varrho}{\varrho} \right)\end{aligned}\quad (11)$$

is the Stokes operator.

Suppose now that the scalar η is replaced by the tensor

$$\eta \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & e \\ 0 & e & 1 \end{pmatrix}, \quad \eta = \text{const}$$

in cylindrical components, then the vector $\eta \text{curl } \mathbf{B}$ is replaced by the vector

$$\frac{\eta}{\varrho} \begin{pmatrix} -f_\zeta \\ \Delta_* F + f_\varrho e \\ e \Delta_* F + f_\varrho \end{pmatrix}.$$

Thus the vector equation (3) yields the three component equations

$$-\frac{\eta}{\varrho} f_\zeta + U_\varrho - \frac{1}{\varrho^2} (G_\varrho f - F_\varrho g) = 0, \quad (12)$$

$$\frac{\eta}{\varrho} (\Delta_* F + f_\varrho e) - \frac{1}{\varrho^2} (G_\zeta F_\varrho - G_\varrho F_\zeta) = 0, \quad (13)$$

$$\frac{\eta}{\varrho} (e \Delta_* F + f_\varrho) + U_\zeta - \frac{1}{\varrho^2} (G_\zeta f - F_\zeta g) = 0. \quad (14)$$

Similarly to the case of helical symmetry [12]–[15] with isotropic η , the e -term in (13) represents a coupling between the poloidal and toroidal parts of the magnetic field. If this term were zero, the boundary conditions would not admit non-trivial solutions. Combination of (12)–(14) leads to

$$\begin{aligned}\frac{\eta}{\varrho} (\nabla f \cdot \nabla F - f \Delta_* F + e F_\varrho \Delta_* F - e f_\varrho f) \\ + U_\zeta F_\varrho - U_\varrho F_\zeta = 0,\end{aligned}\quad (15)$$

$$\begin{aligned}\frac{\eta}{\varrho} (f_\zeta F_\varrho - f_\varrho F_\zeta - e F_\zeta \Delta_* F) - \nabla U \cdot \nabla F \\ + \frac{f}{\varrho^2} \nabla G \cdot \nabla F - \frac{1}{\varrho^2} |\nabla F|^2 g = 0.\end{aligned}\quad (16)$$

Equations (13), (15) and (16) in the fluid can be considered as three equations for the five unknown scalar functions F, f, G, g, U . Outside the conducting fluid

the equations are

$$\Delta_* F = 0, \quad (17)$$

$$f = G = g = \Delta U = 0, \quad (18)$$

and on the discontinuity surface the conditions

$$F, \nabla F, f, G, U \text{ continuous on } S \quad (19)$$

have to be satisfied.

Rather than prescribing the scalars G, g of the velocity field and solving for F, f, U it is easier to solve the inverse dynamo problem [16] by prescribing F, f . This inverse problem can here be formulated as follows: Prescribe F such that it is zero for $\varrho=0, \varrho^2+\zeta^2 \rightarrow \infty$, solves (17) in the external region, and has exactly one minimum in the conducting region such that the curves $F=\text{const}$ in the poloidal plane are non-intersecting. This can be done by, for instance, choosing for the external region the field of a magnetic dipole at $\varrho=\zeta=0$ and continuing F into the interior so that it has exactly one minimum. A second example for choosing F could be the field of a current-carrying circular wire with finite thickness. Such configurations have nested toroidal magnetic surfaces [17] $F=\text{const}$ and the magnetic axis at the minimum of F . Because of

$$G_\zeta F_\varrho - G_\varrho F_\zeta = \varrho (\nabla F \times \nabla \theta) \cdot \nabla G = -|\nabla F| \frac{dG}{dl},$$

where l is the arc length in the poloidal plane along the curve $F=\text{const}$, (13) can be considered as an inhomogeneous first-order ODE for G on the closed curve $F=\text{const}$. The necessary and sufficient condition that G be a single-valued solution of (13) leads to

$$\oint_{F=\text{const}} \eta \varrho (\Delta_* F + f_\varrho e) \frac{dl}{|\nabla F|} = 0. \quad (20)$$

Analogously, the condition that (15) has a single-valued solution U is

$$\oint_{F=\text{const}} \frac{\eta}{\varrho} (\nabla f \cdot \nabla F - f \Delta_* F + e F_\varrho \Delta_* F - e f_\varrho f) \frac{dl}{|\nabla F|} = 0. \quad (21)$$

Thus satisfying the integral relations (20), (21) is necessary and sufficient for the existence of single-valued solutions G, g, U of (12)–(14).

Large Aspect Ratio

That the conditions (20) and (21) can be satisfied is demonstrated for the large-aspect-ratio case. Let

$f, \Delta_* F, G, g$ be non-zero only inside a torus with large radius R and circular cross-section with radius r . Then toroidal coordinates s, ϕ, z are introduced by [18]

$$\varrho = R + x, \quad x = s \cos \phi,$$

$$\theta = -R^{-1} z,$$

$$\zeta = y, \quad y = s \sin \phi.$$

The interior of the torus is described by

$$0 \leq s \leq r < R.$$

In the limit of small inverse aspect ratio $\varepsilon = r/R$ the axisymmetry reduces the plane symmetry. In order that the fields stay finite in this limit, it is useful to scale the scalars by

$$F = RH, \quad f = -Rh, \quad G = RK, \quad g = -Rk.$$

Then (12)–(14) become

$$\eta h_y + U_x + h K_x - k H_x = 0, \quad (22)$$

$$\eta(e \Delta H - h_x) + U_y + h K_y - k H_y = 0, \quad (23)$$

$$\eta(\Delta H - e h_x) + K_x H_y - K_y H_x = 0. \quad (24)$$

Let the function H be chosen such that it depends only on s (cylindrically symmetric magnetic surfaces). Then with s, ϕ as independent variables, (22)–(24) yield

$$\eta \left(e \Delta H \sin \phi + \frac{1}{s} h_\phi \right) + U_s + h K_s - k H' = 0, \quad (25)$$

$$\eta(e \Delta H \cos \phi - h_s) + \frac{1}{s} U_\phi + \frac{1}{s} h K_\phi = 0, \quad (26)$$

$$\eta \left[\Delta H - e \left(h_s \cos \phi - \frac{1}{s} f_\phi \sin \phi \right) \right] - \frac{1}{s} H' K_\phi = 0, \quad (27)$$

where the prime denotes the derivative with respect to s and

$$\Delta H = H'' + \frac{1}{s} H'.$$

The existence conditions (20), (21) read

$$\int_0^{2\pi} \eta \left[\Delta H - e \left(h_s \cos \phi - \frac{1}{s} h_\phi \sin \phi \right) \right] d\phi = 0, \quad (28)$$

$$\int_0^{2\pi} \eta \left[-h_s H' + h \Delta H + e H' \cos \phi \Delta H - e h \left(h_s \cos \phi - \frac{1}{s} h_\phi \sin \phi \right) \right] d\phi = 0. \quad (29)$$

Here, it is seen that (28) is the condition for (27) to have a solution K periodic in ϕ , and (29) is the condition for (26) to have a periodic solution U .

For the case that e does not depend on ϕ the ansatz,

$$\left. \begin{aligned} h &= h_1(s) \cos \phi, \quad K = K_2(s) \sin 2\phi, \\ k &= k_1(s) \sin \phi + k_3(s) \sin 3\phi, \\ U &= U_1(s) \sin \phi + U_3(s) \sin 3\phi \end{aligned} \right\} \quad (30)$$

reduces the system (25)–(27) of PDE's to the system

$$\eta \left[-\frac{1}{2} e \left(h'_1 + \frac{1}{s} h_1 \right) + \Delta H \right] = 0, \quad (31)$$

$$-\frac{1}{2} \eta e \left(h'_1 - \frac{1}{s} h_1 \right) - \frac{2}{s} H' K_2 = 0, \quad \Rightarrow K_2, \quad (32)$$

$$\eta(-h'_1 + e \Delta H) + \frac{1}{s} U_1 + \frac{1}{s} h_1 K_2 = 0, \quad \Rightarrow U_1, \quad (33)$$

$$\frac{3}{s} U_3 + \frac{1}{s} h_1 K_2 = 0, \quad \Rightarrow U_3, \quad (34)$$

$$\eta \left(-\frac{1}{s} h_1 + e \Delta H \right) + U'_1 + \frac{1}{2} h_1 K'_2 - k_1 H' = 0, \quad \Rightarrow k_1, \quad (35)$$

$$U'_3 + \frac{1}{2} h_1 K'_2 - k_3 H' = 0, \quad \Rightarrow k_3 \quad (36)$$

of ODE's. Note that insertion of (30) into (28) yields (31), while (29) is satisfied identically. The system (31) to (36) is explicit in the following sense. Suppose that H, h_1 satisfy (31). The if the sequence (32)–(36) is kept, the system is algebraic in the rest of the dependent variables.

These solutions satisfying (30) have the property that the toroidal flux vanishes, which means that all field lines are closed.

The behaviour at $s \rightarrow 0$ and $s \rightarrow r$ now has to be discussed. Suppose there is a solution of (31) with the analytical property

$$h_1 \rightarrow O(s^1), \quad H' \rightarrow O(s^1), \quad s \rightarrow 0. \quad (37)$$

Then the bracket in (32) is $O(s^2)$ and

$$K_2 \rightarrow O(s^2). \quad (38)$$

Because of (31), the bracket in (33) is $O(s^2)$. It thus follows that

$$U_1 \rightarrow O(s^3), \quad U_3 \rightarrow O(s^3). \quad (39)$$

This, together with the property that the bracket in (35) is $O(s^2)$ leads to

$$k_1 \rightarrow O(s^1), \quad k_3 \rightarrow O(s^3). \quad (40)$$

If one wants to embed the torus in a singly connected conductor without altering the fields, it is required [19] that at $s=r$ there be no surface charge. This can be achieved by

$$\begin{aligned} h_1 &\rightarrow O[(r-s)^3], & H' &\rightarrow O[(r-s)^0], \\ \Delta H &\rightarrow O[(r-s)^2], & s &\rightarrow r. \end{aligned} \quad (41)$$

Equations (32)–(36) then yield

$$\left. \begin{aligned} K_2 &\rightarrow O[(r-s)^2], \\ U_1 &\rightarrow O[(r-s)^2], & U_3 &\rightarrow O[(r-s)^5], \\ k_1 &\rightarrow O[(r-s)^1], & k_3 &\rightarrow O[(r-s)^4] \end{aligned} \right\} \quad (42)$$

which means that the fields \mathbf{v} , $\text{curl } \mathbf{B}$, and ∇U vanish on the toroidal surface $s=r$, from which in turn it follows that there is no electric field outside the torus. So, the torus can be embedded in a singly connected conductor without altering the electromagnetic fields.

Now, the existence problem has been reduced to the question whether (31) has regular solutions with the properties (37) and (41). Writing (31) in the form

$$H'' + \frac{1}{s} H' = \frac{1}{2} e \left(h'_1 + \frac{1}{s} h_1 \right), \quad (43)$$

$$H' = \begin{cases} \frac{1}{2} e_0 C s [r^2(r^2 - s^2)^3 + e_2 s^2 (\frac{1}{2} r^6 - 2 r^4 s^2 + \frac{9}{4} r^2 s^4 - \frac{4}{5} s^6)], & s < r, \\ H'(r)(r/s), & s > r. \end{cases} \quad (45)$$

it is seen that this is not possible for constant e , because for constant e

$$H' = \frac{1}{2} e h_1? \quad (44)$$

is the only regular solution of (43), which would mean that the poloidal field is not seen outside the torus. However, a slight modification of constant e admits solutions having all properties. Consider

$$h_1 = \begin{cases} C s (r^2 - s^2)^3, & s < r, \\ 0, & s > r, \end{cases} \quad e = e_0(r^2 + e_2 s^2).$$

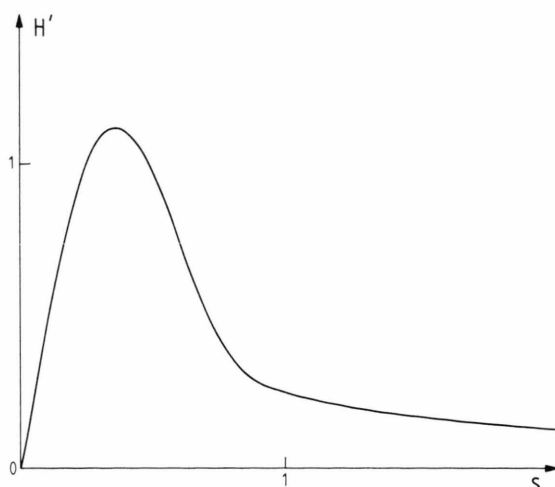


Fig. 1. The function H' defined in (45) for $C=10$, $r=1$, $e_0=1$, $e_2=-1$.

One then has

$$\frac{1}{2} \left(h'_1 + \frac{1}{s} h_1 \right) = \begin{cases} C (r^2 - s^2)^2 (r^2 - 4 s^2), & s < r, \\ 0, & s > r, \end{cases}$$

and

An example of the function H' is shown in Figure 1.

If the anisotropy becomes small, $e \rightarrow 0$, then the externally visible field H stays finite and $h \rightarrow O(e^{-1})$, $K \rightarrow O(e^0)$, $U \rightarrow O(e^{-1})$, $k \rightarrow O(e^{-1})$. A question which has not been discussed here is what physical effects could produce such an anisotropy of the diffusivity η . One possibility is inhomogeneities in the temperature distribution [20].

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